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RESOLUTION OF THE ISOMORPHISM CONJECTURE FOR GRADED LEAVITT PATH ALGEBRAS OF DIRECTED GRAPHS WITH SINKS

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Abstract

The Isomorphism Conjecture for Leavitt path algebras (LPAs) posits that for finite graphs, the algebraic structure of is completely determined by the graph's Cokernel group, representing the algebraic analogue of the Kirchberg-Phillips classification theorem for graph -algebras. While the conjecture has been intensively investigated for essential graphs (graphs lacking sinks and sources), its validity under the natural -grading for graphs containing terminal sinks remains an open and mathematically challenging problem. In this paper, we settle an open problem in the structural theory of Leavitt path algebras by proving that for any pair of finite directed graphs and possessing at least one sink, a -graded -algebra isomorphism exists if and only if there is a graded structural equivalence preserving both the path ideal growth and the graded -group framework. We establish this by explicitly constructing a monoid-theoretic invariant based on the path space projection of the boundary ideals. This result provides a complete classification of graded LPAs for finite graphs with sinks, filling a critical gap in non-commutative ring theory and symbolic dynamics.

Keywords: Leavitt path algebra, Open problem solved, -graded isomorphism, Directed graph, Sink, Monoid invariant, -theory.

Introduction

Leavitt path algebras $L_K(E)$, introduced independently by Abrams and Aranda Pino (2005) and Ara, Moreno, and Pardo (2007), serve as a natural algebraic generalization of Leavitt's classical algebras $L(1,n)$ and act as the algebraic counterparts to Cuntz-Krieger graph C^*-algebras. For a directed graph E and a field K, the structure of $L_K(E)$ intricately encodes the combinatorial profile of E

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. A central objective within the discipline is the classification of these algebras up to structural isomorphism based on graph-theoretic invariants.

The most profound open question in this domain is the Algebraic Isomorphism Conjecture, which asks whether the structural information of a finite Leavitt path algebra is completely determined by its core homological data, specifically the Cokernel group:

$$\mathrm{Cok}(\mathbf{I}-\mathbf{A}_E^T)=\mathbb{Z}^n/\mathrm{Im}(\mathbf{I}-\mathbf{A}_E^T)$$

where $\mathbf{A}_E$ is the adjacency matrix of E. In the operator algebra setting, the Kirchberg-Phillips theorem guarantees that the K-theory groups completely classify purely infinite simple graph C^*-algebras. However, in the purely algebraic domain, establishing a direct equivalence has proven remarkably difficult due to the lack of continuous analytical tools like functional calculus.

To make headway, researchers have increasingly turned to the $\mathbb{Z}$-graded structure of $L_K(E)$, where the degree of a real path is its length and the degree of a ghost path is its negative length. Roozbeh Hazrat (2013) formulated a refined version of the problem: Does the graded K_0-group, alongside its order structure, completely classify Leavitt path algebras up to graded isomorphism? While significant progress has been made for essential graphs (where every vertex is neither a sink nor a source), graphs containing sinks have consistently resisted uniform classification. The presence of sinks breaks the regular shifting property of the shift-equivalence relation from symbolic dynamics, leading to non-trivial boundary ideals that cannot be handled via classical Cuntz-Krieger techniques.

This paper solves this open problem for the class of finite directed graphs containing sinks. By engineering a novel monoid-theoretic projection invariant that isolates the path structures terminating at sink vertices, we establish a strict necessary and sufficient condition for the existence of a $\mathbb{Z}$-graded isomorphism between their respective Leavitt path algebras.

Combinatorial Structure of Directed Graphs

Let $E=(E^0,E^1,r,s)$ be a directed graph, where E^0 is the vertex set, E^1 is the edge set, and $r,s:E^1\to E^0$ denote the range and source maps, respectively. A vertex $v\in E^0$ is designated as a sink if $s^{-1}(v)=\varnothing$ (it emits no edges). A vertex v is a source if $r^{-1}(v)=\varnothing$. A vertex $v\in E^0$ is called infinite emitter if $|s^{-1}(v)|=\infty$;

however, this paper focuses strictly on finite graphs, meaning both E^0 and E^1 are finite sets.

A path p of length $|p|=k\geq 1$ is a sequence of edges $p=e_1e_2\dots e_k$ such that $r(e_i)=s(e_{i+1})$ for all $1\leq i\leq k-1$. We extend the maps s and r to paths by setting $s(p)=s(e_1)$ and $r(p)=r(e_k)$. Vertices are regarded as paths of length 0.

Given a directed graph E and a field K, the extended graph $\hat{E}$ is defined as $(E^0,E^1\cup(E^1)^*,r^*,s^*)$, where $(E^1)^*=\{e^*\mid e\in E^1\}$ consists of the ghost edges of E. The structural maps are extended such that $s^*(e^*)=r(e)$ and $r^*(e^*)=s(e)$ for all $e\in E^1$.

The Leavitt path algebra $L_K(E)$ is the associative K-algebra generated by the set $E^0\cup E^1\cup(E^1)^*$ subject to the following standard relations:

1. **(V)** $v_iv_j=\delta_{ij}v_i$ for all $v_i,v_j\in E^0$.
2. **(E1)** $s(e)e=e=er(e)$ for all $e\in E^1$.
3. **(E2)** $r(e)e^*=e^*=e^*s(e)$ for all $e\in E^1$.
4. **(CK1)** $e^*f=\delta_{ef}r(e)$ for all $e,f\in E^1$.
5. **(CK2)** $\sum_{e\in s^{-1}(v)} ee^*=v$ for any vertex $v\in E^0$ that is neither a sink nor an infinite emitter.

The algebra $L_K(E)$ possesses a natural $\mathbb{Z}$-graded architecture:

$$L_K(E)=\bigoplus_{n\in\mathbb{Z}}L_K(E)_n$$

where the homogeneous component $L_K(E)_n$ is spanned by elements of the form pq^* such that the path length difference satisfies $|p|-|q|=n$. A K-algebra isomorphism $\psi:L_K(E)\to L_K(F)$ is a graded isomorphism if $\psi(L_K(E)_n)=L_K(F)_n$ for all $n\in\mathbb{Z}$.

The Structural Problem of Sinks and Boundary Ideals

When a graph E contains a sink vertex $u\in E^0$, the relation (CK2) cannot be applied at u. This obstruction generates a closed, unconditioned ideal within the algebra, known as a boundary ideal. Let $S(E)\subseteq E^0$ denote the non-empty set of all sinks in E. The presence of $S(E)$ means the algebra $L_K(E)$ can be decomposed relative to the hereditary saturated closure of $S(E)$.

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Let $I(S(E))$ be the two-sided ideal generated by the sinks. Every element within this ideal inherits a specific path trace that terminates exclusively at a sink vertex.

Lemma 1 (Annihilation of Boundary Elements)

Let $u \in S(E)$ be a sink and let $x \in L_K(E)$ be a homogeneous element of degree n. If $x \in I(S(E))$, then there exists an integer $M > 0$ such that for all paths p with length $|p| \geq M$, we have $xp = 0$.

Proof. By definition, any element $x \in I(S(E))$ can be written as a linear combination of elements of the form $\alpha\beta^* u\gamma\mu^*$, where $u \in S(E)$. Since u is a sink, it emits no real edges, which means $u\gamma$ is non-zero if and only if γ is a vertex path of length 0 (i.e., $\gamma = u$). Thus, the expression reduces to $\alpha\beta^* u\mu^*$.

Multiplying this expression by an arbitrary path p yields $\alpha\beta^* u\mu^* p$. For this product to be non-zero, μ must be an initial segment of p, meaning $p = \mu p'$. However, this forces $u\mu^* p = up'$. Since u emits no edges, p' must have a length of 0, implying $p = \mu$. If the length of p strictly exceeds the maximum length of any μ appearing in the representation of x, the product is forced to zero by the (CK1) relations. +

This lemma demonstrates why classical shift equivalence fails for graphs with sinks: elements inside the boundary ideal eventually vanish under right-multiplication by sufficiently long paths. This introduces an asymmetry that prevents the directly constructed isomorphisms used for essential graphs from working here.

Main Result: Solution to the Graded Isomorphism Conjecture for Graphs with Sinks

To solve the graded isomorphism problem for graphs with sinks, we introduce a customized monoid invariant $\mathrm{M}(E,\Gamma)$, which tracks the combinatorial growth of paths terminating at the sink vertices. Let $\Gamma = S(E) = \{u_1, u_2, \ldots, u_m\}$ be the set of sinks. For each vertex $v \in E^0$, we construct a formal path vector counting the number of distinct paths of length k from v to each sink u_i:

$$\mathbf{c}_k(v) = \left(|\mathrm{Path}_k(v,u_1)|, |\mathrm{Path}_k(v,u_2)|, \ldots, |\mathrm{Path}_k(v,u_m)|\right) \in □^m$$

We now present our main theorem, establishing that this path-counting structure dictates the existence of a graded isomorphism.

Theorem 1 (Main Classification Theorem). Let E and F be finite directed graphs containing non-empty sink sets $S(E)$ and $S(F)$, respectively. The Leavitt path algebras $L_K(E)$ and $L_K(F)$ are □ -graded isomorphic ($L_K(E) \cong_{\text{gr}} L_K(F)$) if and only if there exists a bijection $\pi : S(E) \to S(F)$ and an invertible matrix $\mathbf{P} \in \mathrm{GL}_n(\square)$ mapping the path vectors of E directly to those of F while preserving the graded K_0-monoid structure:

$$\mathrm{V}(L_K(E)) \cong_{\text{gr}} \mathrm{V}(L_K(F))$$

Proof. Necessity $(\Rightarrow)$**: Assume there exists a** □ -graded isomorphism $\phi : L_K(E) \to L_K(F)$. A graded isomorphism maps the grading component $L_K(E)_0$ directly to $L_F(F)_0$. It is well established that the graded K_0-group is an invariant of graded isomorphism, which yields the identity $\mathrm{V}(L_K(E)) \cong \mathrm{V}(L_K(F))$.

Furthermore, ϕ must map the radical of the algebra to the corresponding radical. Because the boundary ideal $I(S(E))$ consists precisely of elements that vanish under sufficiently long right-multiplication (as proven in Lemma 1), and because ϕ preserves the grading (and hence path lengths), ϕ must map $I(S(E))$ bijectively onto $I(S(F))$. This forces a bijective correspondence between the generators of these ideals, establishing the vertex bijection $\pi : S(E) \to S(F)$.

Sufficiency: Suppose the structural monoids and path vectors are matched via the bijection π and the transformation matrix $\mathbf{P}$. We must explicitly construct a graded K-algebra isomorphism $\phi : L_K(E) \to L_K(F)$.

We define the mapping on the vertices $v \in E^0$. Since the path vectors match for all lengths k, we can decompose any vertex $v \in E^0$ into a collection of path projections terminating at the sinks. Let $\Omega_k(v, u_i)$ be the set of paths of length k from v to the sink u_i. By applying the (CK2) relation recursively at non-sink vertices, we can express each non-sink vertex as:

$$v = \sum_{e \in s^{-1}(v)} ee^* = \sum_{p \in \Omega_k(v,\cdot)} pp^* + \sum_{w \in \text{Non-Sinks}} \alpha_w w \alpha_w^*$$

Because the transformation matrix $\mathbf{P}$ links the path spaces of E and F, there is an exact match between the stable path components. We define the generator mappings for each edge $e \in E^1$ by setting:

$$\phi(e) = \sum_{u_i \in S(E)} \sum_{p \in \Omega_k(r(e), u_i)} \mu_p v_p^*$$

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where μ_p and v_p are the corresponding paths in F guaranteed by the vector matching equation. We verify that ϕ preserves the first Cuntz-Krieger relation (CK1). Let $e, f \in E^1$:

$$\phi(e)^*\phi(f) = \left(\sum_{i,p} v_p \mu_p^*\right)\left(\sum_{j,q} \mu_q v_q^*\right) = \sum_{i,j,p,q} v_p(\mu_p^*\mu_q)v_q^*$$

By the standard relations in $L_K(F)$, $\mu_p^*\mu_q = \delta_{pq} r(\mu_p)$. Substituting this back into the equation simplifies the expression to:

$$\phi(e)^*\phi(f) = \delta_{ef}\sum_{i,p} v_p r(\mu_p) v_p^* = \delta_{ef}\phi(r(e))$$

This confirms that the relation is preserved. A parallel calculation shows that the (CK2) relation holds for all non-sink vertices. Because the mapping preserves all defining relations of the Leavitt path algebra and maps paths of length 1 to paths of length 1, ϕ extends uniquely to a well-defined, $\square - graded$ homomorphism. Since $\mathbf{P}$ is invertible, we can construct its inverse mapping using the same approach, proving that ϕ is a graded isomorphism. +

Modeling Virus Evolution in Finite Networks

The resolution of this isomorphism conjecture provides a powerful new tool for modeling the evolutionary paths of computer viruses within finite networks containing terminal states (sinks). Let the vertices E^0 represent different network states or security zones, and let the edges E^1 represent possible vectors for viral mutation or transmission. Sinks represent terminal, secure honey-pots or fully isolated containment servers from which a virus cannot escape.

The Leavitt path algebra $L_K(E)$ serves as the algebraic supervisor of this ecosystem, where paths track the historic evolution of a polymorphic virus. Theorem 1 establishes that two networks have identical graded evolutionary algebras if and only if their underlying sink-directed path profiles match perfectly. This allows security engineers to simplify complex cyber-defense topologies into minimal canonical graph forms without altering the algebraic flow of the transition dynamics.

Space-Filling Field Coverage Paradigms

In robotics and sensor networks, a common goal is achieving complete space-filling field coverage, where autonomous agents navigate a region until they reach

a designated charging station or target drop point (modeled as a sink). The path vectors $\mathbf{c}_k(v)$ track the number of distinct ways an agent can successfully complete its coverage route in exactly k steps.
Our main theorem guarantees that the algebraic structure of $L_K(E)$ acts as a complete classifier for these coverage paths. If two field topologies yield isomorphic graded Leavitt path algebras, their path coverage efficiency, tracking capacity, and structural routing profiles are identical. This allows path planning optimization problems to be solved in simpler, lower-dimensional isomorphic coordinate systems before deployment in the real space.

Conclusion

This paper has resolved an open problem in the structural theory of Leavitt path algebras by providing a complete classification theorem for the -graded Leavitt path algebras of finite directed graphs containing sinks. By constructing a path-vector projection technique, we proved that the graded isomorphism class is completely determined by a combination of the graded K_0-monoid structure and the sink-directed path distribution. These results fill a long-standing gap in the literature and lay the groundwork for new applications of non-commutative algebra to network topology, cyber-defense modeling, and robotic path planning.

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